Time Complexity and the divide and conquer strategy

Or : how to measure algorithm run-time
And : design efficient algorithms

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Basic preliminary considerations

- We are interested by the *asymptotic time complexity* T(n) with n being the size of the input
- order of magnitude : $O(f(n))$
  - ➢ $\exists$ A, $\exists$ α $\forall$n>A $g(n) < \alpha f(n)$ => g is said to be $O(f(n))$
  - ➢ Examples :
    - n^2 is $O(n^3)$ (why?), $1000n + 10^{10}$ is $O(n)$

Understanding order of magnitude

If 1000 steps/sec, how large can a problem be in order to be solved in :

Time complexity	1 sec	1 min	1 day
$\log_2 n$	2^{1000}	∞	∞
n	1000	60 000	$8{,}6 \, . \, 10^7$
$n \log n$	140	4893	$5{,}6 \, . \, 10^5$
n^2	31	244	9300
n^3	10	39	442
2^n	10	15	26

Is it worth to improve the code?

- If moving from n^2 to $n.\log n$, definitively
- If your step is running 10 times faster,
  - ➢ For the same problem, 10 time faster!
  - ➢ For the same time how larger might the data be:
    - Linear : 10 time larger
    - $n.\log n$: almost 10 time larger
    - n^2 : 3 time larger
    - 2^n : initial size + 3.3

◄**Forget about**

Complexity of an algorithm

- Depends on the data :
  - If an array is already sorted, some sorting algorithms have a behavior in $O(n)$,
- Default definition : complexity is the complexity in *the worst case*
- Alternative :
  - Complexity in the best case (no interest)
  - Complexity on the average :
    - Requires to define the distribution of the data.

Complexity of a problem

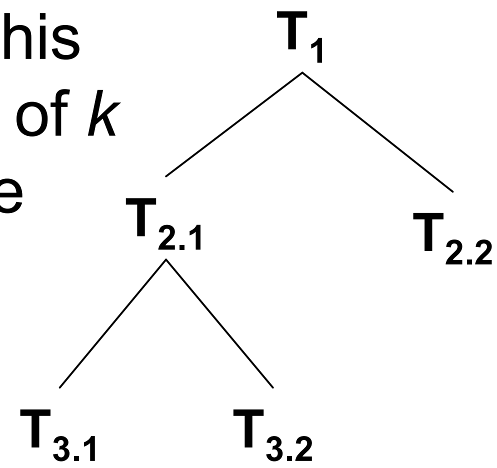
- The complexity of the best algorithm for providing the solution
  - Often the complexity is linear: you need to input the data;
  - Not always the case : the dichotomy search is in $O(n \log n)$ if the data are already in memory
- Make sense only if the problem can be solved :
  - Unsolvable problem : for instance: deciding if a program will stop (linked to what is mathematically undecidable)
  - Solvable problem: for instance: deciding if the maximum of a list of number is positive; complexity $O(n)$

Complexity of sorting

- Finding the space of solutions : one of the permutations that will provide the result sorted : size of the space : n!
- How to limit the search solution
  - Each answer to a test on the data specifies a subset of possible solutions
  - In the best case, the set of possible solution in cut into 2 half

Sorting (cont.)

- If we are smart enough for having this kind of tests : we need a sequence of *k* tests to reach a subset with a single solution.



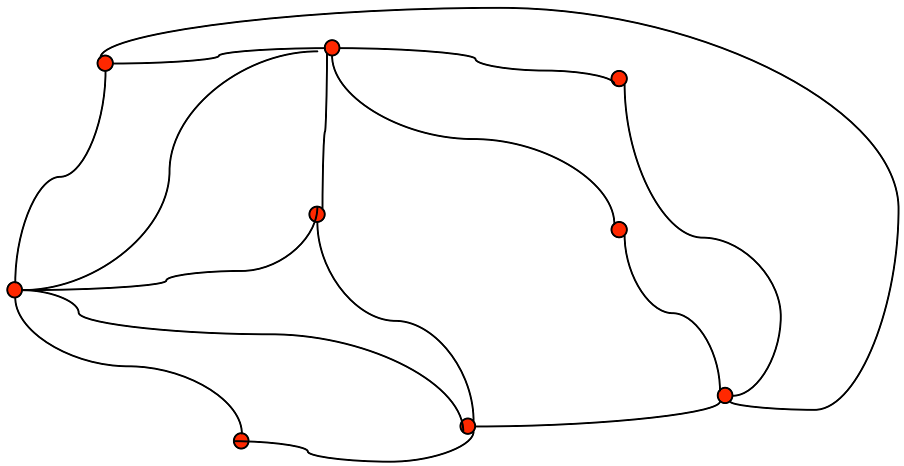
- Therefore : $2^k \sim n!$
- So

$$k \approx \log_2 !n \approx \log_2 \sqrt{2\pi n}\,\frac{n^n}{e^n} \approx \log_2 \sqrt{2\pi n} + n\log_2 n - n\log_2 e$$

- Therefore sorting is at best in $O(n.\log n)$
- And we know an algorithm in $O(n\log n)$

Examples of complexity

- Polyomial sum : $O(n)$
- Product of polynoms : $\cancel{O(n^2)}$? $O(n\log n)$
- Graph coloring : probably $O(2^n)$



- Are 3 colors for a planar graph sufficient?
- Can a set of numbers be splitted in 2 subsets of equal sum?

Space complexity

- Complexity in space : how much space is required?
  - ➢don't forget the stack when recursive calls occur
  - ➢Usually much easier than time complexity

The divide and conquer strategy

- A first example : sorting a set *S* of values
  - sort (*S*) =

```
if |S| ≤ 1 then return S
else divide (S, S1, S2)
     fusion (sort (S1), sort (S2))
end if
```

fusion is linear is the size of its parameter;
divide is either in O(1) or O(*n*)

The result is in *O*(*n*log*n*)

The divide and conquer principle

- General principle :
  - Take a problem of size *n*
  - Divide it into *a* sub problems of size *n/b*
  - this process adds some linear complexity *cn*
- What is the resulting complexity?

$$T(n) = aT\left(\frac{n}{b}\right) + cn$$

$$T(1) = 1$$

- Example . Sorting with fusion ; a=2, b=2

Fundamental complexity result for the divide and conquer strategy

- If $T(n) = aT(\frac{n}{b}) + cn$

  $T(1) = 1$

- Then
  - If a=b : $T(n) = O(n.\log n)$ ◄ Most frequent case
  - If a<b and c>0 : $T(n) = O(n)$
  - If a<b and c=0 : $T(n) = O(\log n)$
  - If a>b :

    $$T(n) = O(n^{\log_b a})$$

Proof : see lecture notes section 12.1.2

Proof steps

- Consider $n = b^k$ $(k = \log_b n)$
- $$\begin{aligned} T(n) &= aT(\frac{n}{b}) + cn \\ aT(\frac{n}{b}) &= a^2T(\frac{n}{b^2}) + a\frac{cn}{b} \\ \ldots & \quad \ldots \\ a^iT(\frac{n}{b^i}) &= a^{i-1}T(\frac{n}{b^{i+1}}) + a^i\frac{cn}{b^i} \\ a^{\log_b(n)}T(1) &= a^{\log_b(n)} \end{aligned}$$
- Summing terms together :

$$T(n) = cn\sum_{i=1}^{k-1}(\frac{a}{b})^i + a^k$$

Proof steps (cont.)

$$T(n) = cn\sum_{i=1}^{k-1}(\frac{a}{b})^i + a^k$$

- a<b ➔ the sum is bounded by a constant and $a^k < n$, so $T(n) = O(n)$
- a=b, c>0 ➔ $a^k = n$, so $T(n) = O(n.\log n)$
- a>b : the (geometric) sum is of order a^k/n
  - Both terms in a^k
  - Therefore $T(n) = O(n^{\log_b a})$

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Application: matrix multiplication

- Standard algorithm
  - For all (i,j) $c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$ O(n³)
- Divide and conquer:
  - Direct way :

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A2 \\ A_{21} & A_{22} \end{pmatrix} \times \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

Counting : b=2, a=8

therefore $O(n^3)$!!!

- Smart implementation: Strassen, able to bring it down to 7
  - Therefore $O(n^{\log_2 7}) = O(n^{2,81})$

Only for large value of *n* (>700)

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Greedy algorithms : why looking for

- A standard optimal search algorithm:

Computes the best solution extending a partial solution S' only if its value exceeds the initial value of *Optimal_Value;*

The result in such a case is *Optimal_S;* these global variables might be modified otherwise

```
Search (S: partial_solution) :
    if Final(S) then if value(S)> Optimal_Value then
                            Optimal_Value := value(S); Optimal_S := S;
                        end if;
    else for each S' extending S loop
                Search (S');
    end if
```

Complexity : if k steps in the loop, if the search depth is n : $O(k^n)$

Instantiation for the search of the longest path in a graph

```
Longest (p: path)
-- compute the longest path without circuits in a graph
--      only if the length extends the value of The_Longest set
--      before the call; in this case Long_Path  is the value of this path, …..
if Cannot_Extend(p) and then length(p)> The_Longest
            then The_Longest := length(p); Long_Path := p;
 else let x be the end node of p;
        for each edge (x,y) such that y ∉ p loop
            Longest (p ⊕ y);
end if;
-- initial call : The_Longest := -1;
              Longest (path (departure_node));
```

Alternative

- Instead of the best solution, a *not too bad* solution?

```
Greedy_search(S: partial_solution) :
  if final (S) then sub_opt_solution := S
  else select the best S' expending S
       greedy_search (S')
  end if;
```

Complexity : $O(n)$

Greedy search for the longest path

```
Greedy_Longest (p: path) :
  if Cannot_Extend(p) then Sub_Opt_Path := p
  else let x be the end node of p;
       select the longest edge (x,y) such that y ∉ p
  exp
       Greedy_Longest (p ⊕ y);
  end if;
```

Obviously don't lead to the optimal solution in the general case

Exercise : build an example where it leads to the worst solution.

How good (bad?) is such a search?

- Depends on the problem
  - Can lead to the worst solution in some cases
  - Sometimes can guarantee the best solution

Example : the minimum spanning tree (find a subset of edges of total minimum cost connecting a graph)

```
Edge_set := ∅
for i in 1..n-1 loop
        Select the edge e with lowest cost not connecting already
        connected nodes
        Add e to Edge_set
    End loop;
```

- Notice that this algorithm might not be in $O(n)$ as we need to find a minimum cost edge, and make sure that it don't connect already connected nodes
  - This can be achieved in log$n$ steps, but is out of scope of this lecture : see the "union-find" data structure in Aho-Hopcroft-Ulman

Conclusion: What to remember

- Complexity on average might differ from worst case complexity : smart analysis required
- For unknown problems, explore first the size of solution space
- *Divide and conquer* is an efficient strategy (exercises will follow); knowing the complexity theorem is required
- Smart algorithm design is essential: a computer 100 times faster will never defeat an exponential complexity