Time Complexity and the divide and conquer strategy

Or: how to measure algorithm run-time
And: design efficient algorithms

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Basic preliminary considerations

• We are interested by the asymptotic time complexity $T(n)$ with $n$ being the size of the input

• order of magnitude : $O(f(n))$
  
  $\exists A, \exists \alpha \forall n > A \ g(n) < \alpha f(n) \Rightarrow g$ is said to be $O(f(n))$

  Examples :
  
  $n^2$ is $O(n^3)$ (why?),  $1000n + 10^{10}$ is $O(n)$
Understanding order of magnitude

If 1000 steps/sec, how large can a problem be in order to be solved in:

<table>
<thead>
<tr>
<th>Time complexity</th>
<th>1 sec</th>
<th>1 min</th>
<th>1 day</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\log_2 n$</td>
<td>$2^{1000}$</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$n$</td>
<td>1000</td>
<td>60,000</td>
<td>$8.6 \cdot 10^7$</td>
</tr>
<tr>
<td>$n \log n$</td>
<td>140</td>
<td>4893</td>
<td>$5.6 \cdot 10^5$</td>
</tr>
<tr>
<td>$n^2$</td>
<td>31</td>
<td>244</td>
<td>9300</td>
</tr>
<tr>
<td>$n^3$</td>
<td>10</td>
<td>39</td>
<td>442</td>
</tr>
<tr>
<td>$2^n$</td>
<td>10</td>
<td>15</td>
<td>26</td>
</tr>
</tbody>
</table>
Is it worth to improve the code?

• If moving from $n^2$ to $n \cdot \log n$, definitively
• If your step is running 10 times faster,
  ➢ For the same problem, 10 time faster!
  ➢ For the same time how larger might the data be:
    ▪ Linear : 10 time larger
    ▪ $n \cdot \log n$ : almost 10 time larger
    ▪ $n^2$ : 3 time larger
    ▪ $2^n$ : initial size + 3.3 ✅ Forget about
Complexity of an algorithm

• Depends on the data:
  ➢ If an array is already sorted, some sorting algorithms have a behavior in $O(n)$,

• Default definition: complexity is the complexity in the worst case

• Alternative:
  ➢ Complexity in the best case (no interest)
  ➢ Complexity on the average:
    ▪ Requires to define the distribution of the data.
Complexity of a problem

• The complexity of the best algorithm for providing the solution
  ➢ Often the complexity is linear: you need to input the data;
  ➢ Not always the case: the dichotomy search is in $O(n \log n)$ if the data are already in memory

• Make sense only if the problem can be solved:
  ➢ Unsolvable problem: for instance: deciding if a program will stop (linked to what is mathematically undecidable)
  ➢ Solvable problem: for instance: deciding if the maximum of a list of number is positive; complexity $O(n)$
Complexity of sorting

• Finding the space of solutions: one of the permutations that will provide the result sorted: size of the space: \( n! \)

• How to limit the search solution
  - Each answer to a test on the data specifies a subset of possible solutions
  - In the best case, the set of possible solution in cut into 2 half
If we are smart enough for having this kind of tests: we need a sequence of $k$ tests to reach a subset with a single solution.

Therefore: $2^k \sim n!$

Therefore sorting is at best in $O(n \cdot \log n)$

And we know an algorithm in $O(n \log n)$
Examples of complexity

- Polynomial sum: $O(n)$
- Product of polynomials: $O(n^2)$ vs. $O(n \log n)$
- Graph coloring: probably $O(2^n)$

- Are 3 colors for a planar graph sufficient?
- Can a set of numbers be split into 2 subsets of equal sum?
Space complexity

• Complexity in space: how much space is required?
  ➢ don’t forget the stack when recursive calls occur
  ➢ Usually much easier than time complexity
The divide and conquer strategy

• A first example: sorting a set $S$ of values
  
  $\text{sort } (S) =$
  
  if $|S| \leq 1$ then return $S$
  else divide ($S$, $S_1$, $S_2$)
      $\text{fusion } (\text{sort } (S_1), \text{sort } (S_2))$
  end if

  $\text{fusion}$ is linear in the size of its parameter;
  $\text{divide}$ is either in $O(1)$ or $O(n)$

  The result is in $O(n \log n)$
The divide and conquer principle

• General principle:
  ➢ Take a problem of size $n$
  ➢ Divide it into $a$ sub problems of size $n/b$
  ➢ this process adds some linear complexity $cn$

• What is the resulting complexity?

$$T(n) = aT\left(\frac{n}{b}\right) + cn$$

$$T(1) = 1$$

• Example. Sorting with fusion; $a=2$, $b=2$
Fundamental complexity result for the divide and conquer strategy

- If \( T(n) = aT\left(\frac{n}{b}\right) + cn \)

  \[ T(1) = 1 \]

- Then
  - If \( a=b \): \( T(n) = O(n\log n) \)
  - Most frequent case
  - If \( a<b \) and \( c>0 \): \( T(n) = O(n) \)
  - If \( a<b \) and \( c=0 \): \( T(n) = O(\log n) \)
  - If \( a>b \):

    \[ T(n) = O\left(n^{\log_b a}\right) \]

**Proof:** see lecture notes section 12.1.2
Proof steps

• Consider $n = b^k$ ($k = \log_b n$)

• $T(n) = aT\left(\frac{n}{b}\right) + cn$

  $aT\left(\frac{n}{b}\right) = a^2T\left(\frac{n}{b^2}\right) + a \frac{cn}{b}$

  $\ldots$  $\ldots$

  $a^iT\left(\frac{n}{b^i}\right) = a^{i-1}T\left(\frac{n}{b^{i+1}}\right) + a^i \frac{cn}{b^i}$

  $a^{\log_b(n)}T(1) = a^{\log_b(n)}$

• Summing terms together:

  $$T(n) = cn \sum_{i=1}^{k-1} \left(\frac{a}{b}\right)^i + a^k$$
Proof steps (cont.)

\[ T(n) = cn \sum_{i=1}^{k-1} \left( \frac{a}{b} \right)^i + a^k \]

• \(a < b\) \(\Rightarrow\) the sum is bounded by a constant and \(a^k < n\), so \(T(n) = O(n)\)

• \(a = b, c > 0\) \(\Rightarrow\) \(a^k = n\), so \(T(n) = O(n \log n)\)

• \(a > b\) : the (geometric) sum is of order \(a^k/n\)

  - Both terms in \(a^k\)
  - Therefore \(T(n) = O(n^{\log_b a})\)
Application: matrix multiplication

- Standard algorithm
  - For all \((i,j)\)
  \[
  c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j} \quad O(n^3)
  \]

- Divide and conquer:
  - Direct way:
    \[
    \begin{pmatrix}
    C_{11} & C_{12} \\
    C_{12} & C_{22}
    \end{pmatrix}
    =
    \begin{pmatrix}
    A_{11} & A_{21} \\
    A_{21} & A_{22}
    \end{pmatrix}
    \times
    \begin{pmatrix}
    B_{11} & B_{12} \\
    B_{21} & B_{22}
    \end{pmatrix}
    \]
    Counting: \(b=2, a=8\)
    therefore \(O(n^3)\) !!!

- Smart implementation: Strassen, able to bring it down to 7
  - Therefore \(O(n^{\log_2 7}) = O(n^{2.81})\)

Only for large value of \(n (>700)\)
Greedy algorithms : why looking for

• A standard optimal search algorithm:
Computes the best solution extending a partial solution $S'$ only if its value exceeds the initial value of $Optimal\_Value$;
The result in such a case is $Optimal\_S$; these global variables might be modified otherwise

```plaintext
Search (S: partial_solution):
  if Final(S) then if value(S) > Optimal_Value then
    Optimal_Value := value(S); Optimal_S := S;
    end if;
  else for each $S'$ extending $S$ loop
    Search ($S'$);
  end if
```

Complexity : if $k$ steps in the loop, if the search depth is $n$ : $O(k^n)$
Instantiation for the search of the longest path in a graph

\[\text{Longest} \ (p: \text{path})\]

-- compute the longest path without circuits in a graph
-- only if the length extends the value of \text{The\_Longest} set
-- before the call; in this case \text{Long\_Path} is the value of this path, 

\[
\text{if Cannot\_Extend}(p) \ \text{and then length}(p) > \text{The\_Longest} \\
\quad \text{then } \text{The\_Longest} := \text{length}(p); \text{Long\_Path} := p; \\
\text{else let } x \text{ be the end node of } p; \\
\quad \text{for each edge } (x, y) \text{ such that } y \notin p \text{ loop} \\
\quad \quad \text{Longest} \ (p \oplus y); \\
\text{end if;}
\]

-- initial call : \text{The\_Longest} := -1;

\[\text{Longest} \ (\text{path (departure\_node)});\]
Alternative

• Instead of the best solution, a *not too bad* solution?

\[
\text{Greedy\_search}(S: \text{partial\_solution}) :
\]
\[
\text{if} \ \text{final} (S) \ \text{then} \ \text{sub\_opt\_solution} := S
\]
\[
\text{else} \ \text{select the best} \ S' \ \text{expending} \ S
\]
\[
\ \text{greedy\_search} \ (S')
\]
\[
\text{end if};
\]

Complexity : $O(n)$
Greedy search for the longest path

\textbf{Greedy\_Longest} (p: path):
\begin{itemize}
\item if \texttt{Cannot\_Extend}(p) then \texttt{Sub\_Opt\_Path} := p
\item else let \(x\) be the end node of \(p\);
\quad select the longest edge \((x, y)\) such that \(y \notin p\)
\quad exp
\quad \textbf{Greedy\_Longest} (p \oplus y);
\end{itemize}
\texttt{end if};

Obviously don't lead to the optimal solution in the general case

Exercise: build an example where it leads to the worst solution.
How good (bad?) is such a search?

• Depends on the problem
   Can lead to the worst solution in some cases
   Sometimes can guarantee the best solution

Example: the minimum spanning tree (find a subset of edges of total minimum cost connecting a graph)

```
Edge_set := ∅  
for i in 1..n-1 loop  
    Select the edge e with lowest cost not connecting already connected nodes  
    Add e to Edge_set  
End loop;
```
• Notice that this algorithm might not be in $O(n)$ as we need to find a minimum cost edge, and make sure that it don’t connect already connected nodes

  ➢ This can be achieved in $\log n$ steps, but is out of scope of this lecture: see the “union-find” data structure in Aho-Hopcroft-Ulman
Conclusion:
What to remember

• Complexity on average might differ from worst case complexity: smart analysis required
• For unknown problems, explore first the size of solution space
• *Divide and conquer* is an efficient strategy (exercises will follow); knowing the complexity theorem is required
• Smart algorithm design is essential: a computer 100 times faster will never defeat an exponential complexity